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Abstract

We derive an expression for the Spearman rank correlation of bivariate scale mixtures of normals (SMN) and we show that within this class, for any value of the correlation parameter, the Spearman rank correlation of the normal is the greatest in absolute value. We then provide expressions for the symmetric generalized hyperbolic, the Bessel, and the Laplace distributions. We further derive an expression for the Spearman rank correlation of the Student t distribution in terms of an easily computable one-dimensional integral, and we also consider the special case of the Cauchy. Finally, we show how our results can be used in a rank-based estimation of the parameters of the Student t distribution.

Keywords: Rank correlation, Rank-based estimation, Scale mixture of normals, Student t, Spearman's rho.
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1. Introduction

The rank correlations of the bivariate normal distribution are well-known, see [18] and [7], who attributes the result to [31]. The Kendall and Spearman rank correlations of the bivariate normal with correlation r are, respectively,

$$\tau_N(r) = \frac{2}{\pi}\arcsin(r), \quad \rho_N(r) = \frac{6}{\pi}\arcsin\left(\frac{r}{2}\right).$$

As shown by [19], the relation between Kendall's tau and the linear correlation parameter holds more generally for all elliptical distributions with continuous marginals, including the bivariate Student t. However, as shown by [16], this invariance does not hold for Spearman's rho, which, to the best of our knowledge is not known in closed form for scale mixtures of normals, an important subclass of the elliptical distributions, even for a very common distribution such as the Student t (for a general discussion of dependence in meta-elliptical distributions, see [1, 10]).

In this paper we first express the Spearman rank correlation of bivariate scale mixtures of normals (SMNs, see [2]) as an expectation of a correlation mixture of the Spearman rank correlation of the Gaussian. We show that within this class, for any value of the correlation parameter, the Spearman rank correlation of the normal is the greatest in absolute value. We then provide expressions for the symmetric generalized hyperbolic, the Bessel, and the Laplace distributions. We further derive an expression for the Spearman rank correlation of the Student t distribution in terms of an easily computable one-dimensional integral, and we also consider the special case of the Cauchy. Finally, we show how our results can be used in a rank-based estimation of the parameters of the Student t distribution.

2. Scale mixtures of normals

Scale mixtures of normals are a subclass of the elliptical distributions. A bivariate scale mixture of normals $\boldsymbol{X} \sim SMN_2(\boldsymbol{\mu}, \boldsymbol{\Sigma}, f_W)$ can be expressed as the product of a normal vector $\boldsymbol{Z} \sim \mathcal{N}_2(0, \boldsymbol{\Sigma})$, and the square root of a

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scalar-valued random variable W with positive support and density f_W, stochastically independent of $\boldsymbol{Z}$:

$$\boldsymbol{X} = \boldsymbol{\mu} + \sqrt{W}\boldsymbol{Z},$$

where $\mathcal{N}_n(0, \boldsymbol{\Sigma})$ is the n-dimensional normal distribution with mean 0 and variance-covariance $\boldsymbol{\Sigma}$ and $\boldsymbol{\mu}$ is a vector location parameters. Conditionally on W, $\boldsymbol{X}$ is Gaussian:

$$\boldsymbol{X}|W = w \sim \mathcal{N}_2(\boldsymbol{\mu}, w\boldsymbol{\Sigma}).$$

The density of $\boldsymbol{X}$ then follows by integration with respect to the density of the mixing variable W:

$$f_{\boldsymbol{X}}(x_1, x_2) = \int_0^\infty f_{\boldsymbol{X}|W}(x_1, x_2|w) f_W(w) dw,$$

where $f_{\boldsymbol{X}|W}$ is the density of the bivariate normal.

By Sklar's theorem (see [32]) the joint cumulative distribution function $F_{\boldsymbol{X}}$ of the bivariate SMN variable $\boldsymbol{X}$ can be expressed in terms of a copula C, which captures the underlying dependence structure, and of its marginals F_i, $i \in \{1, 2\}$:

$$F_{\boldsymbol{X}}(x_1, x_2) = C\{F_1(x_1), F_2(x_2)\}.$$

The Kendall and Spearman rank correlations, τ and ρ, depend on the copula but not on the marginal distributions (see, e.g., [26]):

$$\tau = 4 \int_0^1 \int_0^1 C(u_1, u_2) dC(u_1, u_2) - 1, \quad \rho = 12 \int_0^1 \int_0^1 C(u_1, u_2) du_1 du_2 - 3.$$

Thus we limit attention to the case $\boldsymbol{X} \sim SMN_2(0, \boldsymbol{P}, f_W)$ when $\boldsymbol{Z}$ follows a standardized normal with unit variances, correlation matrix $\boldsymbol{P}$, and correlation r. Whenever W has a finite mean, $\boldsymbol{X}$ has finite second moments, and thus r is the correlation of $\boldsymbol{X}$. Throughout the paper, we will refer to r as the correlation parameter, even when $\boldsymbol{X}$ does not have finite second moments.

A number of well-known distributions can be written as scale mixtures of normals. The most general distribution we consider is the bivariate symmetric generalized hyperbolic distribution $\boldsymbol{X} \sim GH_2(\lambda, \chi, \psi, \boldsymbol{P})$, which depends on the correlation parameter r through the matrix $\boldsymbol{P}$, a parameter λ that influences tail behavior, a scaling parameter χ and a shape parameter ψ. It is a special case of the multivariate generalized hyperbolic distribution introduced in [3, 4] and it has been also referred to as the generalized multivariate modified Bessel (see [33]). Its density is

$$f_{\boldsymbol{X}}(\boldsymbol{x}) = |\boldsymbol{P}|^{-1/2} \frac{(\sqrt{\chi\psi})^{-\lambda}\psi}{2\pi K_\lambda(\sqrt{\chi\psi})} \frac{K_{\lambda-1}\left\{\sqrt{\psi(\chi + \boldsymbol{x}^\top \boldsymbol{P}^{-1}\boldsymbol{x})}\right\}}{\left\{\sqrt{\psi(\chi + \boldsymbol{x}^\top \boldsymbol{P}^{-1}\boldsymbol{x})}\right\}^{(1-\lambda)}},$$

where $K_\lambda(z)$ is a modified Bessel function of the third kind:

$$K_\lambda(z) = \frac{1}{2} \int_0^\infty y^{\lambda-1} \exp\left\{-\frac{1}{2} z(y + y^{-1})\right\} dy.$$

It can be written as a scale mixture of normals when W follows a generalized inverse Gaussian, $W \sim GIG(\lambda, \chi, \psi)$, with density

$$f_W(w) = \frac{\chi^{-\lambda}\left(\sqrt{\chi\psi}\right)^\lambda}{2K_\lambda\left(\sqrt{\chi\psi}\right)} w^{\lambda-1} \exp\left\{-\frac{1}{2}\left(\chi w^{-1} + \psi w\right)\right\}.$$

The bivariate generalized hyperbolic distribution includes as special cases the hyperbolic when $\lambda = 3/2$ (see [5]), the normal inverse gamma (NIG) when $\lambda = -1/2$, and the bivariate Bessel distribution of [22] (the univariate Bessel dates back to [23]), also known as variance-gamma or generalized Laplace, with parameter q when $\lambda = q + 1, \chi = 0$, and where, in order to have unit variances, we further impose $\psi = 2(q + 1)$. The density of the bivariate Bessel

$\boldsymbol{X} \sim \mathcal{B}_2(q, \boldsymbol{P})$ is

$$f_X(\boldsymbol{x}) = \frac{1}{\pi}|\boldsymbol{P}|^{-1/2}\frac{q+1}{\Gamma(1+q)2^q}\left\{\sqrt{2(q+1)\boldsymbol{x}^\top\boldsymbol{P}^{-1}\boldsymbol{x}}\right\}^q K_q\left\{\sqrt{2(q+1)\boldsymbol{x}^\top\boldsymbol{P}^{-1}\boldsymbol{x}}\right\}.$$

It can be written as a scale mixture of normals when $W \sim \Gamma(q+1, q+1)$, where the density of a Gamma variable $\Gamma(\alpha,\beta)$ is

$$f_W(w) = \frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}\exp(-\beta w).$$

When we further impose $q = 0$, we obtain the bivariate symmetric Laplace distribution $\boldsymbol{X} \sim \mathcal{L}_2(\boldsymbol{P})$, with correlation matrix $\boldsymbol{P}$, which corresponds to $W \sim \Gamma(1,1)$, or alternatively a mean one exponential $W \sim \mathscr{E}(1)$. Its density is

$$f_X(\boldsymbol{x}) = \frac{|\boldsymbol{P}|^{-1/2}}{\pi}K_0\left(\sqrt{2\boldsymbol{x}^\top\boldsymbol{P}^{-1}\boldsymbol{x}}\right).$$

The generalized hyperbolic distribution also includes the Pearson type VII (see [11]) when $\lambda = -\nu/2$ and $\psi = 0$, whose density is

$$f_X(\boldsymbol{x}) = \frac{1}{2\pi}\sqrt{\frac{\nu}{\chi}}|\boldsymbol{P}|^{-1/2}\left(1+\frac{\boldsymbol{x}^\top\boldsymbol{P}^{-1}\boldsymbol{x}}{\chi}\right)^{-(\nu/2+1)}.$$

It can be written as a scale mixture of normals when $W \sim IG(\nu/2, \chi/2)$, and the density of the inverse Gamma $IG(\alpha,\beta)$ is

$$f_W(w) = \frac{\beta^\alpha}{\Gamma(\alpha)}w^{-(\alpha+1)}\exp(-\beta/w).$$

When $\lambda = -\nu/2$, $\chi = \nu$ and $\psi = 0$, $W \sim IG(\nu/2, \nu/2)$, it reduces to the Student t with degrees of freedom ν, $\boldsymbol{X} \sim T_2(\boldsymbol{P}, \nu)$, whose density is

$$f_X(\boldsymbol{x}) = \frac{1}{2\pi}|\boldsymbol{P}|^{-1/2}\left(1+\frac{\boldsymbol{x}^\top\boldsymbol{P}^{-1}\boldsymbol{x}}{\nu}\right)^{-(\nu/2+1)}.$$

With unit degrees of freedom $\nu = 1$, $W \sim IG(1/2, 1/2)$ and we obtain the bivariate Cauchy, $\boldsymbol{X} \sim \mathscr{C}_2(\boldsymbol{P})$:

$$f_X(\boldsymbol{x}) = \frac{1}{2\pi}|\boldsymbol{P}|^{-1/2}\left(1+\boldsymbol{x}^\top\boldsymbol{P}^{-1}\boldsymbol{x}\right)^{-3/2}.$$

3. Spearman's rho of scale mixtures of normals (SMN)

Proposition 1 (Spearman's rho for scale mixture of normals). *The Spearman rank correlation of a bivariate scale mixture of normals* $\boldsymbol{X} = \begin{pmatrix}X_1\\X_2\end{pmatrix} \sim SMN(0, \boldsymbol{P}, f_W)$ *is*

$$\rho_{SMN} = \frac{6}{\pi}\mathrm{E}_{\tilde{\mathrm{V}}}\left\{\arcsin\left(\mathrm{r}\tilde{\mathrm{V}}\right)\right\}, \tag{1}$$

where the mixing density is

$$f_{\tilde{V}}(\tilde{v}) = \int_0^\infty\int_{u_1\tilde{v}}^{u_1/\tilde{v}}\frac{4}{(u_1u_2)^4}f_W\left(\frac{\tilde{v}}{u_1u_2}\right)f_W\left(\frac{u_1-u_2\tilde{v}}{u_1u_2^2}\right)f_W\left(\frac{u_2-u_1\tilde{v}}{u_1^2u_2}\right)du_2du_1, \tag{2}$$

when the scaling random variable W has support on $[0,\infty)$. *More generally, when the support of variable W is* $[a,b]$ *with $b > a \geq 0$, integration in (2) is over the domain* $D = \{u_1, u_2 : a \leq \tilde{v}/(u_1u_2) \leq b; a \leq (u_1 - u_2\tilde{v})/(u_1u_2^2) \leq b; a \leq (u_2 - u_1\tilde{v})/(u_1^2u_2) \leq b\}$.

Proof. Let W_i, $i \in \{1,2,3\}$, be three i.i.d. variables with density f_W. Define a bivariate vector $\tilde{\boldsymbol{X}} = \begin{pmatrix}\tilde{X}_1\\\tilde{X}_2\end{pmatrix}$, whose components are two independent SMN variables:

$$\tilde{X}_i|W_i = w_i \sim \mathcal{N}_1(0, w_i), \quad i \in \{1, 2\}.$$

The bivariate SMN vector $\boldsymbol{X}$ can be represented as $\boldsymbol{X}|W_3 = w_3 \sim \mathcal{N}_2(0, w_3\boldsymbol{P})$. Thus $\tilde{\boldsymbol{X}}$ is a bivariate vector of independent SMN variables and $\boldsymbol{X}$ is a bivariate vector of joint SMN variables, independent of $\tilde{\boldsymbol{X}}$. Combining these two vectors into vector $\boldsymbol{\mathcal{X}} = \begin{pmatrix}\tilde{\boldsymbol{X}} \\ \boldsymbol{X}\end{pmatrix}$, we get

$$\boldsymbol{\mathcal{X}}|(W_1 = w_1, W_2 = w_2, W_3 = w_3) \sim \mathcal{N}_4(0, \boldsymbol{\Omega}),$$

and

$$\boldsymbol{\Omega} = \begin{pmatrix}\boldsymbol{\Omega_{11}} & 0 \\ 0 & w_3\boldsymbol{P}\end{pmatrix}, \quad \boldsymbol{\Omega_{11}} = \begin{pmatrix}w_1 & 0 \\ 0 & w_2\end{pmatrix}.$$

Define $\boldsymbol{Y} = \boldsymbol{B_0}\boldsymbol{\mathcal{X}}$, with $\boldsymbol{B_0} = (\boldsymbol{I_2}, -\boldsymbol{I_2})$, where $\boldsymbol{Y}$ is a difference between a bivariate SMN and a bivariate vector of independent SMN variables. Then

$$\boldsymbol{Y}|(W_1 = w_1, W_2 = w_2, W_3 = w_3) \sim \mathcal{N}_2(0, \boldsymbol{B_0\Omega B_0}),$$

where

$$\boldsymbol{B_0\Omega B_0} = \begin{pmatrix}w_1 + w_3 & rw_3 \\ rw_3 & w_2 + w_3\end{pmatrix},$$

and, defining $\tilde{V} = W_3\{(W_1 + W_3)(W_2 + W_3)\}^{-1/2}$, the correlation is $r\tilde{v}$, where $\tilde{v}$ is the realization of $\tilde{V}$. By definition of Spearman's rho (see, e.g., [18]),

$$\begin{aligned}\rho_{SMN} &= 12\Pr(X_1 \le \tilde{X}_1, X_2 \le \tilde{X}_2) - 3 = 12\Pr(Y_1 > 0, Y_2 > 0) - 3 \\ &= 12\mathrm{E}_{\tilde{V}}\left\{\mathrm{E}\left(\mathbb{1}_{\{Y_1>0,Y_2>0\}}|\tilde{V}\right)\right\} - 3 = 12\left[\frac{1}{4} + \frac{1}{2\pi}\mathrm{E}_{\tilde{V}}\left\{\arcsin\left(r\tilde{V}\right)\right\}\right] - 3 = \frac{6}{\pi}\mathrm{E}_{\tilde{V}}\left\{\arcsin\left(r\tilde{V}\right)\right\},\end{aligned}$$

where the second line follows from the definition of $\boldsymbol{Y}$; the third line follows from the law of iterated expectations and the fact that orthant probabilities depend only on correlations when the mean is equal to zero; the fourth line follows from computing the orthant probability of the normal, see, e.g., [9]. Thus we can express Spearman's rho as an expectation, over random variable $\tilde{V}$, of a bivariate normal orthant probability, whose correlation is the only part that depends on $\tilde{V}$. The expression of the mixing density is obtained by considering the trivariate transformation $U_1 = (W_1 + W_3)^{-1/2}$, $U_2 = (W_2 + W_3)^{-1/2}$ and $\tilde{V} = W_3\{(W_1 + W_3)(W_2 + W_3)\}^{-1/2}$ of the independent vector (W_1, W_2, W_3), with inverse transformation $w_1 = (u_1 - u_2\tilde{v})/(u_1u_2^2)$, $w_2 = (u_2 - u_1\tilde{v})/(u_1^2u_2)$ and $w_3 = \tilde{v}/(u_1u_2)$, with Jacobian determinant $4/(u_1u_2)^4$, and integrating out u_1 and u_2 in the range $u_1 \ge 0$ and $u_1\tilde{v} \le u_2 \le u_1/\tilde{v}$, which corresponds to $w_i > 0$ for $i \in \{1, 2, 3\}$. □

Equation (1) shows that the Spearman rank correlation of a scale mixture of normals is an expectation of a correlation mixture of the Spearman rank correlation of the Gaussian, with the mixing density given in (2).

Corollary 1 (**Upper bound of Spearman's rho for scale mixtures of normals**)**.** *Within the class of scale mixtures of normals (SMN), the normal has the largest Spearman rank correlation in absolute value for a given value of the correlation parameter r:* $|\rho_{SMN}| \le |\rho_N|$.

Proof. This follows from Proposition 1, Jensen's inequality and the correlation inequality. Without loss of generality, assume $r > 0$. Given that arcsin is increasing, odd and convex on $[0, 1]$, we get

$$\rho_{SMN} = (6/\pi)\,\mathrm{E}_{\tilde{V}}\left\{\arcsin\left(r\tilde{V}\right)\right\} \le (6/\pi)\arcsin\left\{r\mathrm{E}_{\tilde{V}}\left(\tilde{V}\right)\right\} \le (6/\pi)\arcsin(r/2) = \rho_N,$$

where the first inequality is due to Jensen's inequality and the last inequality follows from

$$\begin{aligned} \mathrm{E}\left(\tilde{\mathrm{V}}\right) &= \mathrm{E}\left(\tilde{\mathrm{V}}_1\tilde{\mathrm{V}}_2\right) = \mathrm{E}\left(\tilde{\mathrm{V}}_1\right)\mathrm{E}\left(\tilde{\mathrm{V}}_2\right) + \mathrm{corr}\left(\tilde{\mathrm{V}}_1, \tilde{\mathrm{V}}_2\right)\left\{\mathrm{var}\left(\tilde{\mathrm{V}}_1\right)\mathrm{var}\left(\tilde{\mathrm{V}}_2\right)\right\}^{1/2} \\ &\leq \mathrm{E}\left(\tilde{\mathrm{V}}_1\right)^2 + \mathrm{var}\left(\tilde{\mathrm{V}}_1\right) = \mathrm{E}\left(\tilde{\mathrm{V}}_1^2\right) = \mathrm{E}\left\{\mathrm{W}_3/(\mathrm{W}_1 + \mathrm{W}_3)\right\} = 1/2, \end{aligned}$$

with $\tilde{V}_i = \{W_3/(W_i + W_3)\}^{1/2}$, for $i \in \{1, 2\}$. The inequality follows from the correlation inequality $\mathrm{corr}\left(\tilde{\mathrm{V}}_1, \tilde{\mathrm{V}}_2\right) \leq 1$ and the fact that $\tilde{V}_1$ and $\tilde{V}_2$ are identically distributed, and the last equality follows from the fact that the W_is are independent and identically distributed. □

Proposition 2 (**Spearman's rho for the generalized hyperbolic distribution**). *The Spearman rank correlation of a bivariate generalized hyperbolic distribution $\boldsymbol{X} \sim GH_2(\lambda, \chi, \psi, \boldsymbol{P})$ is given by (1), with mixing density*

$$f_{\tilde{V}}(\tilde{v}) = \frac{\left(1-\tilde{v}^2\right)^{3\lambda/2}}{\tilde{v}^{\lambda/2+1}} \int_{\tilde{v}/(\tilde{v}^2+1)}^{1/2} \frac{t^{2\lambda}\left\{(1+\tilde{v}^2)t - \tilde{v}\right\}^{-\lambda/2-1}}{(1-4t^2)^{1/2}(1-\tilde{v}t)^{3\lambda/2}} \frac{K_{3\lambda}\left\{\left(\frac{\chi\psi(1-\tilde{v}^2)(1-\tilde{v}t)}{\tilde{v}\{(1+\tilde{v}^2)t-\tilde{v}\}}\right)^{1/2}\right\}}{\left\{K_\lambda(\sqrt{\chi\psi})\right\}^3} dt.$$

Proof. As the generalized hyperbolic distribution is a scale mixture of normals, by Proposition 1, Spearman's rho is given by (1). Upon substitution of the density of the $GIG(\lambda, \chi, \psi)$ into (2), we obtain

$$f_{\tilde{V}}(\tilde{v}) = (C/2)\,\tilde{v}^{\lambda-1} \int_0^\infty \int_{u_1\tilde{v}}^{u_1/\tilde{v}} I(u_1, u_2) du_2 du_1,$$

where

$$C = \frac{(\psi/\chi)^{\frac{3\lambda}{2}}}{K_\lambda\left(\sqrt{\chi\psi}\right)^3}, \quad I(u_1, u_2) = \frac{\{(u_1 - u_2\tilde{v})(u_2 - u_1\tilde{v})\}^{\lambda-1}}{(u_1u_2)^{4\lambda}} \exp\left\{-\frac{h(u_1, u_2)}{2}\right\},$$

and

$$h(u_1, u_2) = \psi\left\{\frac{1}{u_1^2} + \frac{1}{u_2^2} - \frac{\tilde{v}}{(u_1u_2)}\right\} + \chi\left\{\frac{u_1u_2}{\tilde{v}} + \frac{u_1u_2^2}{(u_1 - u_2\tilde{v})} + \frac{u_1^2u_2}{(u_2 - u_1\tilde{v})}\right\}.$$

By symmetry of the domain of integration $D = \{u_1 \geq 0, u_1\tilde{v} \leq u_2 \leq u_1/\tilde{v}\}$ around the 45 degree line $u_2 = u_1$ and symmetry of the integrand $I(u_1, u_2) = I(u_2, u_1)$,

$$f_{\tilde{V}}(\tilde{v}) = C\tilde{v}^{\lambda-1} \int_0^\infty \int_{u_1\tilde{v}}^{u_1} I(u_1, u_2) du_2 du_1.$$

Using the change of variable $(u_1, u_2) = \left\{1/\left(R^{1/2}\cos(\theta)\right), 1/\left(R^{1/2}\sin(\theta)\right)\right\}$ with Jacobian determinant $-1/\left[2\{R\cos(\theta)\sin(\theta)\}^2\right]$, we obtain

$$f_{\tilde{V}}(\tilde{v}) = (C/2)\,\tilde{v}^{\lambda-1} \int_{\arctan(\tilde{v})}^{\pi/4} (\cos(\theta)\sin(\theta))^{2\lambda}\left\{(1+\tilde{v}^2)\cos(\theta)\sin(\theta) - \tilde{v}\right\}^{\lambda-1} \tilde{I}(\theta)d\theta,$$

with

$$\tilde{I}(\theta) = \int_0^\infty R^{\bar{\lambda}-1} \exp\left\{-\frac{1}{2}\left(\bar{\chi}R^{-1} + \bar{\psi}R\right)\right\}dR,$$

where $\bar{\lambda} = 3\lambda$, $\bar{\chi} = \chi\left(1-\tilde{v}^2\right)/\left[\tilde{v}\left\{(1+\tilde{v}^2)\cos(\theta)\sin(\theta) - \tilde{v}\right\}\right]$, and $\bar{\psi} = \psi\{1 - \tilde{v}\cos(\theta)\sin(\theta)\}$. By recognizing the density of the generalized hyperbolic, which integrates to one over the positive real line,

$$\tilde{I}(\theta) = \frac{2K_{\bar{\lambda}}\left(\sqrt{\bar{\chi}\bar{\psi}}\right)}{\bar{\chi}^{-\bar{\lambda}}\left(\sqrt{\bar{\chi}\bar{\psi}}\right)^{\bar{\lambda}}}.$$

Collecting terms and using the final change of variable $t = \cos(\theta)\sin(\theta) = \sin(2\theta)/2$ with $dt = \cos(2\theta)d\theta$ and Jacobian determinant $\left(1-4t^2\right)^{-1/2}$ yields the result. □

Remark 1. The Spearman rank correlation of the GH depends on χ and ψ only through their product. Thus, for example, when $\psi = 0$, regardless of the value of χ, the Spearman rank correlation of the Pearson VII is equal to that of the Student t with degrees of freedom ν. This can be seen also from the fact that $W \sim IG(\nu/2, \chi/2) = (\chi/\nu) IG(\nu/2, \nu/2)$. This leads to the same distribution for $\tilde{V}$, since χ/ν simplifies, and therefore the Spearman rank correlation of the Pearson of type VII is the same as that of the Student t.

Proposition 3 (Spearman's rho for the Bessel distribution). *The Spearman rank correlation of the bivariate Bessel distribution $\boldsymbol{X} \sim \mathcal{B}_2(q, \boldsymbol{P})$ is given by (1), with mixing density*

$$f_{\tilde{V}}(\tilde{v}) = 4\frac{\Gamma(3+3q)}{\Gamma(1+q)^3}\tilde{v}^q \int_{\tilde{v}/(\tilde{v}^2+1)}^{1/2} \frac{t^{2(1+q)}\{(1+\tilde{v}^2)t - \tilde{v}\}^q}{(1-4t^2)^{1/2}(1-\tilde{v}t)^{3(1+q)}} dt.$$

Proof. As the Bessel distribution is a scale mixture of normals, by Proposition 1, Spearman's rho is given by (1). Upon substitution of the density of the $\Gamma(q+1, q+1)$ into (2), the density of $\tilde{V}$ follows from using polar coordinates with radius R and angle θ in the integral of (2), recognizing that the integral involving the radius is proportional to an absolute moment of the normal of order $5+6q$, and further using the change of variable $t = \sin(2\theta)/2$. Alternatively, it can be obtained as a limit case of the mixing density of the generalized hyperbolic when $\chi = 0$ by writing the two Bessel functions $K_\nu(y)$ when y is small as $\Gamma(\nu)2^{\nu-1}y^{-\nu}$ as in [4], using the substitution $\lambda = q+1$ and simplifying. □

Remark 2 (Spearman's rho for the Laplace distribution). The Spearman rank correlation of the bivariate symmetric Laplace distribution $\boldsymbol{X} \sim \mathcal{L}_2(\boldsymbol{P})$, with correlation matrix $\boldsymbol{P}$ is given by (1), with mixing density

$$f_{\tilde{V}}(\tilde{v}) = \frac{4\tilde{v}}{(4-\tilde{v}^2)^2}\left[7 - 8\tilde{v}^2 + \tilde{v}^4 + \frac{2(2+\tilde{v}^2)}{\tilde{v}(4-\tilde{v}^2)^{1/2}} \arctan\left\{\frac{\left(1-\tilde{v}^2\right)\left(4-\tilde{v}^2\right)^{1/2}}{(3-\tilde{v}^2)\,\tilde{v}}\right\}\right].$$

This expression can be found by setting $q = 0$ in the mixing density of the Bessel and solving the integral.

Proposition 4 (Spearman's rho for the Student t). *The Spearman rank correlation, ρ_T, of a bivariate Student t distribution $\boldsymbol{X} \sim T_2(\boldsymbol{P}, \nu)$ is given by (1), with mixing density*

$$f_{\tilde{V}}(\tilde{v}) = 2\frac{\Gamma(\nu)^2\Gamma(3\nu/2)}{\Gamma(\nu/2)^3\Gamma(2\nu)}\tilde{v}^{\nu-1}(1-\tilde{v}^2)^{\nu/2-1}{}_2F_1(\nu,\nu;2\nu,1-\tilde{v}^2), \quad 0 < \tilde{v} < 1, \tag{3}$$

where ${}_2F_1(a,b;c,z) = \Gamma(c)/\{\Gamma(a)\Gamma(c-a)\}\int_0^1 v^{a-1}(1-v)^{c-a-1}(1-vz)^{-b}dv$ is the Gauss hypergeometric function.

Proof. As the Student t distribution is a scale mixture of normals, by Proposition 1, Spearman's rho is given by (1). To compute the density of $\tilde{V}$, we write $\tilde{V} = \tilde{V}_1\tilde{V}_2$, where, for $i \in \{1,2\}$,

$$\tilde{V}_i = \{W_3/(W_i + W_3)\}^{1/2},$$

and $W_i \sim IG(\nu/2, \nu/2)$. We can also write

$$\tilde{V}_i = \{(1/W_i)/(1/W_i + 1/W_3)\}^{1/2},$$

where $1/W_i \sim \Gamma(\nu/2, \nu/2)$ and therefore $(\tilde{V}_1, \tilde{V}_2)$ are square roots of the bivariate beta distribution defined in [28], and the density of $\tilde{V}$ is obtained by further appealing to Theorem 2.1 of [25], who derive the distribution of the product of the components of the bivariate Beta distribution as

$$f_t(t) = \frac{\Gamma(a+b+c)\Gamma(a+c)\Gamma(b+c)}{\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(a+b+2c)}t^{a-1}(1-t)^{c-1}{}_2F_1(a+c, a+c; a+b+2c, 1-t),$$

where in our case $a = b = c = \nu/2$. Finally, (3) is obtained from $f_t(t)$ through $\tilde{V} = T^{1/2}$, with inverse transformation $t = \tilde{v}^2$ and Jacobian determinant $2\tilde{v}$. □

Corollary 2 (**Spearman's rho for Student t when the degrees of freedom** $\nu \to 0$)**.** *In the limit when the degrees of freedom parameter of the Student t tends to zero, Spearman's rho for the Student t tends to the Kendall rank correlation of the normal:*

$$\lim_{\nu \to 0} \rho_T(r, \nu) = \tau_N(r).$$

Proof. The moments of $\tilde{V}$ are obtained from those of the product of the components of a bivariate Beta distribution in [25]:

$$\mathrm{E}\left(\tilde{\mathrm{V}}^{\mathrm{h}}\right) = \frac{\Gamma(\nu)^2 \Gamma(3\nu/2) \Gamma\{(\nu + \mathrm{h})/2\}}{\Gamma(2\nu)\Gamma(\nu/2)^2 \Gamma(\nu + \mathrm{h}/2)} {}_3\mathrm{F}_2(\nu, \nu, \nu/2; 2\nu, \nu + \mathrm{h}/2; 1),$$

where ${}_3F_2(a, b, c; d, e; z)$ is the generalized hypergeometric function, see, e.g., [21]. For any $h > 0$, taking the limit of these moments when $\nu \to 0$,

$$\lim_{\nu \to 0} \mathrm{E}\left(\tilde{\mathrm{V}}^{\mathrm{h}}\right) = \frac{1}{3},$$

since $\lim_{\nu \to 0} \frac{\Gamma(\nu)^2 \Gamma(3\nu/2) \Gamma((\nu+h)/2)}{\Gamma(2\nu)\Gamma(\nu/2)^2 \Gamma(\nu+h/2)} = \frac{1}{3}$ and $\lim_{\nu \to 0} {}_3F_2(\nu, \nu, \nu/2; 2\nu, \nu + h/2; 1) = 1$. A power series expansion of $\arcsin(r\tilde{V})$ in the expression of Spearman's rho for the Student t yields

$$\rho_T(r, \nu) = \frac{6}{\pi} \mathrm{E}_{\tilde{\mathrm{V}}} \left\{\arcsin\left(\mathrm{r}\tilde{\mathrm{V}}\right)\right\} = \frac{6}{\pi} \left\{\mathrm{rE}(\tilde{\mathrm{V}}) + \left(\frac{1}{2}\right) \frac{\mathrm{r}^3}{3} \mathrm{E}(\tilde{\mathrm{V}}^3) + \left(\frac{1}{2}\right)\left(\frac{3}{4}\right) \frac{\mathrm{r}^5}{5} \mathrm{E}(\tilde{\mathrm{V}}^5) + \cdots \right\}.$$

Using the limit of the moments of $\tilde{V}$, and recognizing the series expansion of $\arcsin(r)$, $\lim_{\nu \to 0} \rho_T(r, \nu) = \frac{2}{\pi} \arcsin(r) = \tau_N(r)$. □

We have shown that when the degrees of freedom approach zero, all moments of $\tilde{V}$ are $1/3$. This happens if the density collapses to a Bernoulli with probability $1/3$. If instead we consider the limit when $\nu \to \infty$, the density of $\tilde{V}$ has point mass at $\frac{1}{2}$. This follows by expressing $W_i \sim IG(\nu/2, \nu/2)$ as $W_i = (Q/\nu)^{-1}$, with $Q \sim \chi^2_{[\nu]}$. By the law of large numbers, when $\nu \to \infty$, $Q/\nu \xrightarrow{p} 1$, as Q can be viewed as a sum of ν independent $\chi^2_{[1]}$ variables. By the continuous mapping theorem, $W_i \xrightarrow{p} 1$ and $\tilde{V} = g(W_1, W_2, W_3) \xrightarrow{p} 1/2$ since $g(x, y, z) = z/\sqrt{(x + z)(y + z)}$ is a continuous function. For any $h > 0$, taking the limit when $\nu \to \infty$, the moments of $\tilde{V}$ become powers of $1/2$:

$$\lim_{\nu \to \infty} \mathrm{E}\left(\tilde{\mathrm{V}}^{\mathrm{h}}\right) = \left(\frac{1}{2}\right)^{\mathrm{h}},$$

and the Spearman rank correlation of the Student t is equal to that of the Gaussian.

Remark 3 (Spearman's rho for Cauchy distribution). Whenever $\nu \leq 2$, the Student t has no second order moments and therefore correlation does not exist. However, Spearman's rho can still be computed. For instance when $\nu = 1$, $\boldsymbol{X}$ follows a Cauchy distibution, whose Spearman rank correlation is given by (1), with mixing density

$$f_{\tilde{V}}(\tilde{v}) = -\frac{2}{\pi} \frac{\ln(\tilde{v})}{(1 - \tilde{v}^2)^{3/2}},$$

and when $\nu = 2$, the mixing density is

$$f_{\tilde{V}}(\tilde{v}) = \frac{8\tilde{v}}{(\tilde{v}^2 - 1)^3} \{(\tilde{v}^2 + 1)\ln(\tilde{v}) + 1 - \tilde{v}^2\}.$$

Fig.1 shows the mixing densities $f_{\tilde{V}}(\tilde{v})$ for the Cauchy, the Student t with degrees of freedom $\nu = 2$ and $\nu = 4$, the Bessel with $q = 3$, the Laplace and the generalized hyperbolic with parameters $(\lambda, \sqrt{\chi\psi}) = (-1, 2)$ and $(\lambda, \sqrt{\chi\psi}) = (0, 0.01)$, and Table 1 shows the corresponding values of Spearman's rho for a range of values of the correlation parameter r. The Student t with low degrees of freedom, the Laplace, the Bessel and the generalized hyperbolic with $\lambda = -1$ exhibit only slightly lower vales of Spearman's rho than the normal. On the other hand, the Cauchy and the generalized hyperbolic with $(\lambda, \sqrt{\chi\psi}) = (0, 0.01)$, whose mixing densities have tails with a vertical asymptote at 0 and 1, exhibit values of Spearman's rho that are much farther below those of the normal.

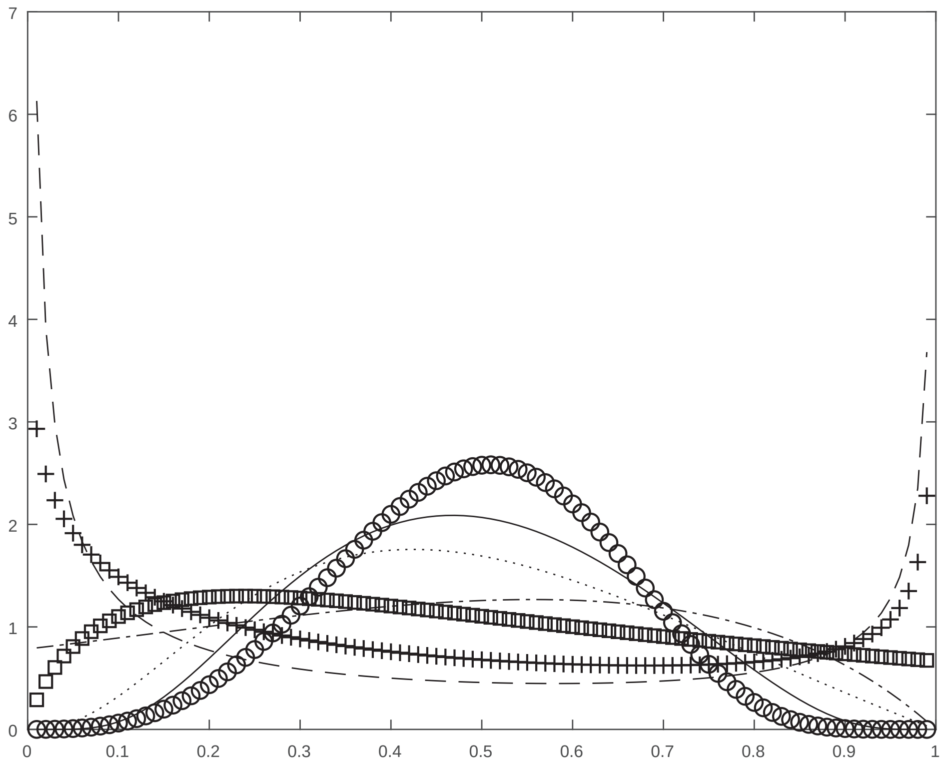

Fig. 1: This figure shows the mixing density $f_{\tilde{V}}(\tilde{V})$ for the Cauchy (plus), the Student t with $\nu = 2$ degrees of freedom (square), the Student t with $\nu = 4$ degrees of freedom (dotted), the Laplace (dash-dotted), the Bessel with $q = 3$ (circles) and the generalized hyperbolic with $(\lambda, \sqrt{\chi\psi}) = (-1, 2)$ (solid) and $(\lambda, \sqrt{\chi\psi}) = (0, 0.01)$ (dashed), respectively.

Table 1:
Spearman rank correlation for different values of the correlation parameter r and for different scale mixture of normal distributions: the normal (second column), the Cauchy (third column), the Student t distribution with $\nu = 2$ and $\nu = 4$ degrees of freedom (fourth and fifth columns), the Laplace (sixth column), the Bessel (seventh column) and the generalized hyperbolic distribution with parameters $(\lambda, \sqrt{\chi\psi}) = (-1, 2)$ and $(\lambda, \sqrt{\chi\psi}) = (0, 0.01)$ (eighth and ninth columns).

r	Normal	Cauchy	Student t		Laplace	Bessel	Generalized hyperbolic	
			$\nu = 2$	$\nu = 4$		$q = 3$	$\lambda = -1$, $\sqrt{\chi\psi} = 2$	$\lambda = 0$, $\sqrt{\chi\psi} = 0.01$
0.1	0.096	0.084	0.089	0.092	0.091	0.094	0.094	0.081
0.2	0.191	0.169	0.179	0.185	0.182	0.189	0.187	0.163
0.3	0.288	0.255	0.270	0.279	0.274	0.284	0.282	0.245
0.4	0.385	0.342	0.361	0.373	0.367	0.380	0.377	0.330
0.5	0.483	0.432	0.455	0.469	0.462	0.477	0.474	0.417
0.6	0.582	0.525	0.552	0.567	0.559	0.576	0.573	0.508
0.7	0.683	0.623	0.651	0.668	0.660	0.677	0.674	0.604
0.8	0.786	0.728	0.757	0.772	0.765	0.780	0.778	0.709
0.9	0.891	0.845	0.870	0.882	0.877	0.888	0.886	0.829

4. A rank-based estimation of the parameters of the Student t

Combined with the expression for Kendall's tau, our expressions for Spearman's rho can be used as a basis for a rank-based estimation method of the correlation and shape parameters of the Student t distribution. Such a method is copula-based, as the only information that it retains from the marginal distributions are the ranks, and rank correlations depend on the copula and not on the marginal densities (see, e.g., [26]). Such a method is especially useful when the degrees of freedom are low. In that case, the method of moments only delivers consistent estimates when the degrees of freedom are larger than 4, which guarantees that kurtosis exists, and moreover, the estimators have finite variance only when the degrees of freedom are larger than 8 (see [30]). Maximum likelihood estimation (MLE) on the other hand suffers from numerical problems and the existence of many local maxima (see [20]). Given that it is based on the ranks of the data and not on their absolute value, the rank-based method should be robust to outliers (see [8, 34]), which also makes it a good candidate starting value for MLE.

The idea of estimating copula parameters from rank correlations is not new. Estimation of the parameter of the Gaussian copula from Spearman's rank correlation was proposed in [17]. Moment estimation of the correlation parameters of the Student t copula by inversion of Kendall's tau, was discussed in [24], but whereas we propose to estimate the degrees of freedom based on Spearman's rho, they use MLE to estimate the degrees of freedom. In [14] the authors discuss what they call a "naive" method for the Archimedean copulas, which consists in estimating the parameter of the copula by inverting Kendall's tau. They trace the idea of inverting rank correlations to estimate copula parameters to [13] for the Frank and [27] for the Clayton.

The rank-based method is based on inversion of the following transformation Υ from parameters (r,ν) to rank correlations (τ,ρ_T):

$$\begin{pmatrix}\tau\\ \rho_T\end{pmatrix} = \Upsilon(r,\nu) = \begin{pmatrix}\frac{2}{\pi}\arcsin(r)\\ \frac{6}{\pi}\mathrm{E}_{\tilde{\mathrm{V}}}\left\{\arcsin\left(\mathrm{r}\tilde{\mathrm{V}}\right)\right\}\end{pmatrix}, \tag{4}$$

in which Kendall's tau depends on the correlation parameter r according to $\tau = (\pi/2)\arcsin(r)$, while Spearman's rho depends both on the correlation parameter r and the degrees of freedom ν, through (1), along with the mixing density given in (3). The rank-based estimation procedure works provided that the inverse of Υ exists, which is true as long as the Jacobian is invertible. Define $\boldsymbol{J}$, the Jacobian of the transformation Υ, whose determinant is $|\boldsymbol{J}| = |\frac{\partial\tau}{\partial r}\frac{\partial\rho_T}{\partial\nu}|$, since $\frac{\partial\tau}{\partial\nu}$=0. Since $\frac{\partial\tau}{\partial r} = (2/\pi)/(1-r^2)^{1/2}$ is always different from zero, the rank-based estimation procedure works, provided that $\frac{\partial\rho_T}{\partial\nu} \neq 0$. When $r = 1$ ($r = -1$), $\tau = 1$ ($\tau = -1$), and by Theorem 5.1.11 of [26], this implies that $\rho_T = 1$ ($\rho_T = -1$). When $r = 0$, $\rho_T = 0$, since $\arcsin(0) = 0$. Thus in these three cases, Spearman's rho does not depend on ν, and therefore ν is not identifiable from ρ_T.

Next, we establish consistency and asymptotic normality of the rank-based estimators of the correlation parameter r and the degrees of freedom ν under the following assumption, which guarantees that Υ^{-1}, the inverse of Υ defined in (4), exists.

Assumption 1. *The Spearman rank correlation of the Student t,* $\rho_T(r,\nu) = \frac{6}{\pi}\mathrm{E}_{\tilde{\mathrm{V}}}\left\{\arcsin\left(\mathrm{r}\tilde{\mathrm{V}}\right)\right\}$ *is strictly increasing (decreasing) in the degrees of freedom* ν *when* $0 < r < 1$ ($-1 < r < 0$).

Consistency of rank-based estimation. *Under Assumption 1, the rank-based estimators of the correlation parameter and the degrees of freedom* $(\hat{r},\hat{\nu})$ *are consistent estimators whenever* $r \in (-1,1)\setminus\{0\}$.

This follows from the fact that, since they are U-statistics, the empirical versions of Kendall's tau, $\hat{\tau}$, and Spearman's rho, $\hat{\rho}$, are consistent estimators of their population counterparts τ and ρ_T (see [29]). Assumption 1 implies that $r\frac{\partial\rho_T}{\partial\nu} > 0$, which establishes that $|\boldsymbol{J}| > 0$, and therefore that Υ^{-1} exists and is differentiable and therefore continuous, provided that $r \in (-1,1)\setminus\{0\}$. The consistency of the rank-based estimators $(\hat{r},\hat{\nu})$ then follows from an application of the continuous mapping theorem.

Asymptotic normality of rank-based estimator. *Under Assumption 1, whenever* $r \in (-1,1)\setminus\{0\}$, *the rank-based estimators* $\hat{r}$ *and* $\hat{\nu}$ *are asymptotically normally distributed:*

$$\sqrt{n}\begin{pmatrix}\hat{r}-r\\ \hat{\nu}-\nu\end{pmatrix} \sim \mathcal{N}_2\left(0,(\boldsymbol{J}^{-1})^{\top}\boldsymbol{\Sigma}\boldsymbol{J}^{-1}\right), \quad \textit{with} \quad \boldsymbol{\Sigma} = \begin{pmatrix}\sigma_\tau^2 & \sigma_{\tau,\rho}\\ \sigma_{\tau,\rho} & \sigma_\rho^2\end{pmatrix},$$

where expressions for σ_τ^2, σ_ρ^2 *and* $\sigma_{\tau,\rho}$ *are given in [15].*

This is a consequence of the fact, shown in [15], that Kendall's tau and Spearman's rho are U-statistics that follow a joint asymptotic normal distribution:

$$\sqrt{n}\begin{pmatrix}\hat{\tau}-\tau\\ \hat{\rho}_T-\rho_T\end{pmatrix}\sim\mathcal{N}_2\left(0,\boldsymbol{\Sigma}\right).$$

Given Assumption 1, Υ^{-1} exists, and since $\partial^{i+j}\Upsilon/\partial r^i\partial\nu^j$ exists for all $i+j\leq 2$, Υ^{-1} is continuous. Under these conditions, the result follows by a standard application of the delta method.

Assumption 1 is supported by extensive numerical simulations, see Fig.2, which shows the values of Spearman's rho for the Gaussian for different values of r, and the corresponding values of Spearman's rho of the Student t for degrees of freedom ν ranging from 0 to 10. For positive values of r, as the degrees of freedom ν increase, the Spearman rank correlation of the Student t converges from below to the value of that of the Gaussian. Since *arcsin* is an odd function, when the correlation parameter r is negative, the sign of the Spearman rank correlation is reversed, and the convergence is from above.

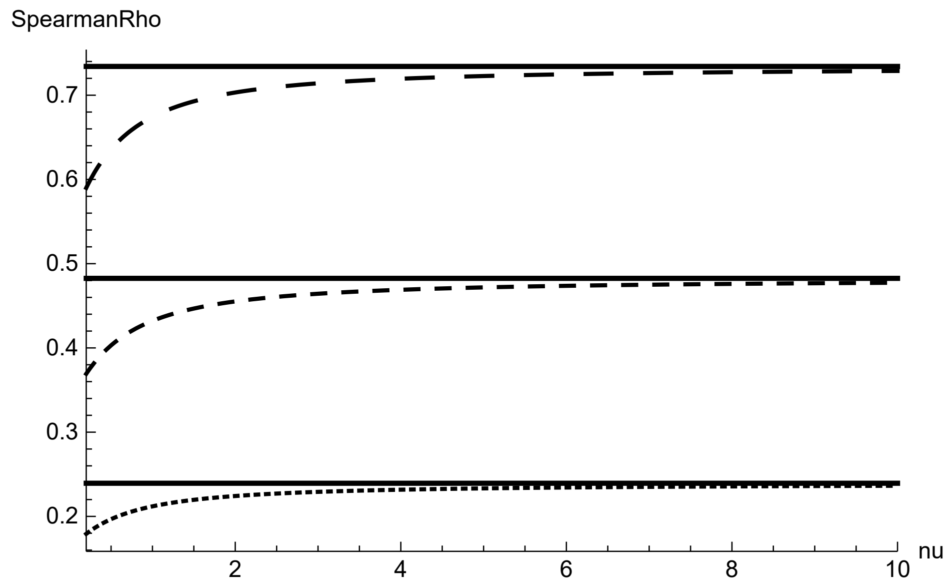

Fig. 2: This figure shows as solid horizontal lines the values of Spearman's rho of the Gaussian for values of the correlation parameter r of 0.25, 0.5 and 0.75, and as dotted lines the corresponding values of Spearman's rho for the Student t with degrees of freedom ν between 0 and 10.

The assumption states that, for a given correlation parameter r, the degrees of freedom parameter ν acts like a dependence parameter. It implies that the tail dependence of the Student t for low degrees of freedom comes at the expense of lower overall dependence, as measured by Spearman's rho. Combined with Corollary 2, the assumption implies that the Spearman rank correlation of the Student t is always greater than Kendall's tau in absolute value. This holds, even though the Student t is neither left-tail decreasing nor right-tail increasing, which are sufficient conditions for Spearman's rho to be greater than Kendall's tau in absolute value (see [6, 12]). More specifically, the Spearman rank correlation of the Student t increases from the value of Kendall's tau to the value of the Spearman rank correlation of the Gaussian, as the degrees of freedom increase from zero to infinity.

We can also show that a sufficient condition for Assumption 1 is the following assumption about the moments of the distribution of $\tilde{V}$, which is also supported by numerical evidence:

Assumption 2. *The odd moments of order $h\geq 3$ of $\tilde{V}$, $\mathrm{E}\left(\tilde{\mathrm{V}}^{\mathrm{h}}\right)$ are strictly decreasing in the degrees of freedom ν.*

To see that Assumption 2 implies Assumption 1, we use a power series expansion of $\arcsin(r\tilde{V})$ in $\rho_T=\frac{6}{\pi}\mathrm{E}_{\tilde{\mathrm{V}}}\left\{\arcsin\left(\mathrm{r}\tilde{\mathrm{V}}\right)\right\}$, and setting $r = 1, \rho_T = 1$, which yields

$$\frac{\pi}{6} = \mathrm{E}\left(\tilde{\mathrm{V}}\right) + \left(\frac{1}{2}\right)\frac{\mathrm{E}\left(\tilde{\mathrm{V}}^3\right)}{3} + \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\frac{\mathrm{E}\left(\tilde{\mathrm{V}}^5\right)}{5} + \cdots$$

Taking derivatives with respect to the degrees of freedom ν,

$$\frac{\partial \mathrm{E}\left(\tilde{\mathrm{V}}\right)}{\partial \nu} + \left(\frac{1}{2}\right)\frac{1}{3}\frac{\partial \mathrm{E}\left(\tilde{\mathrm{V}}^3\right)}{\partial \nu} + \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\frac{1}{5}\frac{\partial \mathrm{E}\left(\tilde{\mathrm{V}}^5\right)}{\partial \nu} + \cdots = 0.$$

Combining this last equation with the expression of the derivative of Spearman's rho with respect to the degrees of freedom ν,

$$\frac{\partial \rho_T(r,\nu)}{\partial \nu} = \frac{6r}{\pi}\left[\left(\frac{1}{2}\right)\frac{1}{3}(r^2-1)\frac{\partial \mathrm{E}\left(\tilde{\mathrm{V}}^3\right)}{\partial \nu} + \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\frac{1}{5}(r^4-1)\frac{\partial \mathrm{E}\left(\tilde{\mathrm{V}}^5\right)}{\partial \nu} + \cdots\right],$$

which is positive (negative), whenever $r > 0$ ($r < 0$), provided Assumption 2 holds. Fig.3 shows the first four odd moments of $\tilde{V}$ across a range of values of the degrees of freedom ν. As can be seen in the figure, in the limit when the degrees of freedom $\nu \to 0$, $\mathrm{E}\left(\tilde{\mathrm{V}}^{\mathrm{h}}\right) = 1/3$ for all $h \geq 0$, whereas, when $\nu \to \infty$, the distribution collapses to a point mass at $1/2$, and therefore $\mathrm{E}\left(\tilde{\mathrm{V}}^{\mathrm{h}}\right) = (1/2)^{\mathrm{h}}$.

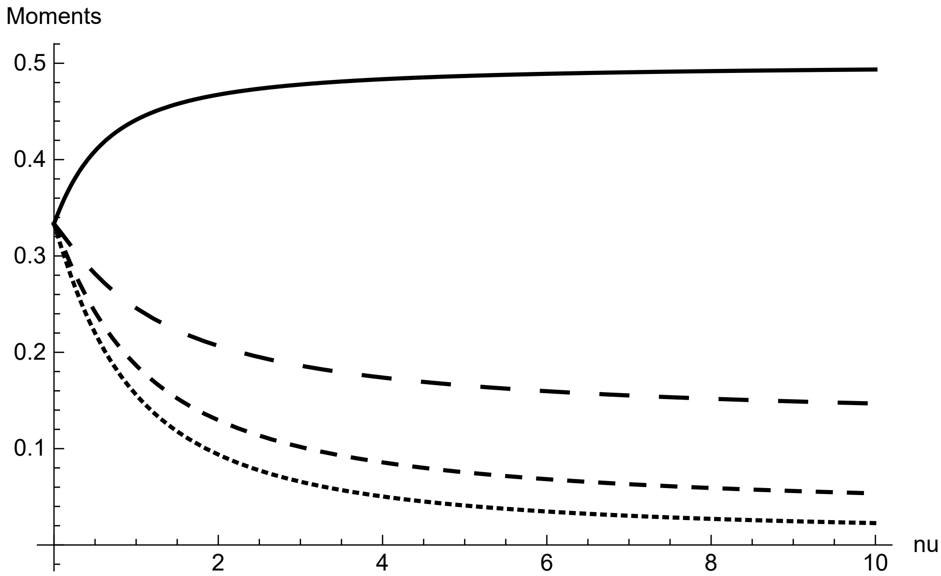

Fig. 3: This figure shows the first four odd moments of $\tilde{\nu}$, $E\left(\tilde{V}^h\right)$ for $h \in \{1,3,5,7\}$, when the degrees of freedom ν vary between 0 and 10. The solid line represents the first moment ($h = 1$), which increases to 0.5 as ν increases, while the dashed lines represent the third, fifth and seventh moment ($h \in \{3,5,7\}$), which decrease to 0.5^h as ν increases.

In order to check how the rank-based procedure works in practice, we conduct a Monte Carlo simulation exercise with 1000 replications and for samples of size 1000 and 10000. Panel A of Table 2 shows the average bias from the rank-based estimation of the correlation parameter r, as well as the degrees of freedom ν of the Student t for the sample size of 1000. Correlation parameters are well-estimated for all possible cases of the Student t. The degrees of freedom parameter of the Student t shows a slight positive bias for small correlations ($r = 0.1$), which decreases for correlation parameters of about 0.2 or 0.3 and seems to flatten out for higher values of r. Although the absolute bias tends to increase with the degrees of freedom, this trend is much less pronounced for relative bias. Panel B shows that when the sample size increases, the bias vanishes for all values of the parameters.

Table 2:
Bias of the rank-based estimation of the parameters of the bivariate Student t distribution for different values of the correlation parameter r, based on a Monte Carlo simulation with 1000 replications and a sample size of 1000 in panel A and 10000 in panel B. We consider the Student t distribution with degrees of freedom $\nu = 1$, $\nu = 2$ and $\nu = 4$. For each distribution we show the bias in the shape parameter, followed by the bias in the correlation parameter.

Student t						
	Panel A: sample size 1000					
r	$\nu = 1$	r	$\nu = 2$	r	$\nu = 4$	r
0.1	0.1176	-0.0011	0.3831	-0.0035	0.6849	0.0007
0.2	0.0136	-0.0002	0.0697	-0.0014	0.1559	-0.0010
0.3	0.0085	-0.0008	0.0118	-0.0003	0.0829	0.0000
0.4	-0.0008	0.0002	0.0175	0.0010	0.0633	0.0001
0.5	0.0054	0.0005	0.0213	0.0006	0.0623	-0.0023
0.6	0.0031	-0.0008	0.0193	-0.0002	0.0743	-0.0003
0.7	0.0068	0.0000	0.0183	0.0003	0.0505	-0.0005
0.8	0.0084	-0.0002	0.0059	0.0001	0.0774	0.0007
0.9	0.0126	-0.0003	0.0432	0.0001	0.1337	-0.0002
	Panel B: sample size 10000					
r	$\nu = 1$	r	$\nu = 2$	r	$\nu = 4$	r
0.1	0.0030	0.0002	0.0109	-0.0009	0.0328	-0.0003
0.2	0.0008	-0.0003	0.0031	0.0001	0.0027	0.0003
0.3	-0.0008	0.0002	0.0032	-0.0002	0.0055	-0.0000
0.4	-0.0017	-0.0004	0.0066	0.0002	0.0160	0.0002
0.5	0.0017	-0.0006	0.0013	-0.0006	-0.0036	-0.0002
0.6	0.0001	-0.0002	-0.0017	-0.0002	0.0150	-0.0001
0.7	0.0016	0.0003	0.0014	-0.0001	-0.0012	0.0002
0.8	0.0009	0.0000	0.0019	-0.0000	0.0087	-0.0002
0.9	0.0015	-0.0000	-0.0028	-0.0001	0.0167	-0.0001

We apply our procedure to five years of daily log-return data for Intel (INTC), Microsoft (MSFT) and General Electric (GE) from 1996 to 2000. This is the same data as in Example 5.57 of [24]. We estimate the correlation and degrees of freedom parameters for each pair of returns using the rank-based estimation procedure and we obtain $(\hat{r}, \hat{\nu}) = (0.594, 8.143), (0.355, 5.110), (0.416, 7.222)$, respectively for the INTC-MSFT, INTC-GE and MSFT-GE pairs. As a comparison, in Example 5.59, [24] obtain correlation parameters of $(0.59, 0.36, 0.42)$ and degrees of freedom of 6.5 in a MLE estimation of a trivariate Student t copula from the ranks of the same data. The correlation parameters are extremely similar across both procedure, and their degrees of freedom are close to the average of the degrees of freedom over the three pairs (6.825).

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